

# A PROBLEM OF RAMANUJAN, ERDŐS AND KÁTAI ON THE ITERATED DIVISOR FUNCTION

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**ABSTRACT.** We determine asymptotically the maximal order of  $\log d(d(n))$ , where  $d(n)$  is the number of positive divisors of  $n$ . This solves a problem first put forth by Ramanujan in 1915.

## 1 Introduction

Let  $d(n)$  denote the number of positive divisors of an integer  $n$ . The extreme large values of  $d(n)$  were studied by Wigert [10], (see also [4, Theorem 432]). Wigert proved that

$$m_1(x) := \max_{n \leq x} \log d(n) \sim (\log 2) \frac{\log x}{\log_2 x}.$$

Here  $\log_k x$  denotes the  $k$ -th iterate of the logarithm. The lower bound comes from considering integers of the form  $N_k = p_1 \cdots p_k$ , where  $p_j$  denotes the  $j$ th smallest prime. Here  $d(N_k) = 2^k$ , while  $\log N_k \sim k \log k$  by the prime number theorem. In his seminal 1915 paper on highly composite numbers [7], Ramanujan gave a more precise asymptotic for  $m_1(x)$ . At the very end of his paper, Ramanujan posed the problem of finding the extreme large values of  $d(d(n))$ . By considering integers of the form

$$(1.1) \quad 2^1 \cdot 3^2 \cdot 5^4 \cdots p_k^{p_k-1},$$

Ramanujan showed that

$$m_2(x) := \max_{n \leq x} \log d(d(n)) \geq (\sqrt{2} \log 4 + o(1)) \frac{\sqrt{\log x}}{\log_2 x}.$$

The problem of finding the order of  $m_2(x)$  has been mentioned in Erdős [1], Ivić [5], and has been mentioned by Ivić in problem sessions in Ottawa [6] and Oberwolfach.

Erdős and Kátaı [3] showed  $m_2(x) = (\log x)^{1/2} (\log_2 x)^{O(1)}$  (see (4.1) on p. 270 of [3]). Twenty years later Erdős and Ivić [2] improved the upper bound to

$$m_2(x) \ll \left( \frac{\log x \log_2 x}{\log_3 x} \right)^{1/2}.$$

Smاتی [8, 9] gave a further improvement

$$m_2(x) \ll \sqrt{\log x},$$

the best estimate known to date. Constructions similar to Ramanujan's seem rather natural, and one might expect that  $m_2(x) \ll \frac{\sqrt{\log x}}{\log_2 x}$ . This is indeed the case, as we now show. More precisely, we prove an asymptotic formula for  $m_2(x)$  with an error term.

**Theorem 1.** *We have*

$$m_2(x) = \frac{\sqrt{\log x}}{\log_2 x} \left( c + O \left( \frac{\log_3 x}{\log_2 x} \right) \right),$$

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where

$$c = \left( 8 \sum_{j=1}^{\infty} \log^2(1 + 1/j) \right)^{1/2} = 2.7959802335 \dots$$

In particular, Theorem 1 implies that

$$\limsup_{n \rightarrow \infty} \frac{\log d(d(n)) \log_2 n}{\sqrt{\log n}} = c.$$

Ramanujan's examples (1.1) are seen to be suboptimal with respect to the constant  $c$ , since  $\sqrt{2} \log 4 = 1.9605 \dots$

There is a closely related problem, to estimate the extreme values of  $\omega(d(n))$ , where  $\omega(n)$  is the number of distinct prime factors of  $n$ . In fact, both Erdős and Ivić [2] and Smati [9] obtained upper bounds for  $d(d(n))$  by first bounding  $\omega(d(n))$  and then using the elementary inequality  $\log d(m) \ll (\log_2 m) \omega(m)$  (see, e.g., Lemme 3.3 of [8] or Lemma 3.2 below). For this problem, Ramanujan's examples (1.1) are essentially optimal, providing the true order and constant in the asymptotic for  $w(x) = \max_{n \leq x} \omega(d(n))$ .

**Theorem 2.** *We have*

$$w(x) = \frac{\sqrt{\log x}}{\log_2 x} \left( \sqrt{8} + O\left(\frac{\log_3 x}{\log_2 x}\right) \right),$$

Previously, Erdős and Ivić [2] had shown

$$w(x) \ll \left( \frac{\log x \log_3 x}{\log_2 x} \right)^{1/2},$$

and later Smati [8] found the true order  $w(x) \ll \frac{\sqrt{\log x}}{\log_2 x}$ .

## 2 The lower bound in Theorem 1

**Notation and basic prime number estimates.** Throughout, we make use of the asymptotic

$$(2.1) \quad p_j = j(\log j + \log_2 j + O(1)),$$

which is a simple consequence of the prime number theorem with error term  $\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$ . Here  $\pi(x)$  is the number of primes which are  $\leq x$ . We also denote by  $\Omega(n)$  the number of prime power divisors of  $n$ .

*Proof of the lower bound in Theorem 1.* Let  $x$  be large and define  $\varepsilon = 10 \frac{\log_3 x}{\log_2 x}$ . Let

$$(2.2) \quad t = \left\lfloor \left( \frac{8 \log 2}{c} - \varepsilon \right) \frac{\sqrt{\log x}}{\log_2 x} \right\rfloor, \quad a_i = \left\lfloor \frac{1}{2^{i/t} - 1} \right\rfloor \quad (1 \leq i \leq t),$$

and let

$$n = (p_1 \cdots p_{a_1})^{p_1-1} (p_{a_1+1} \cdots p_{a_1+a_2})^{p_2-1} \cdots (p_{a_1+\cdots+a_{t-1}+1} \cdots p_{a_1+\cdots+a_t})^{p_t-1}.$$

The Taylor expansion of  $\exp(\frac{\log 2}{t})$  shows that  $a_1 = \lfloor (2^{1/t} - 1)^{-1} \rfloor = t/\log 2 + O(1)$ . By (2.2), for every positive integer  $j$ , there are  $y_j := \lfloor \frac{\log(1+1/j)}{\log 2} t \rfloor$  indices  $i$  with  $a_i \geq j$ . Also,  $a_1 + \cdots + a_t \ll t \log t$ . Using (2.1), we have  $\log p_{a_1+\cdots+a_i} \leq \log t + 2 \log_2 t + O(1)$ , hence

$$\log n \leq \sum_{i=1}^t a_i (p_i - 1) \log p_{a_1+\cdots+a_i} \leq (\log^2 t + 3(\log_2 t) \log t + O(\log t)) \sum_{i=1}^t i a_i.$$

From  $y_j = O(t/j)$  and the definition of  $c$  we obtain

$$(2.3) \quad \begin{aligned} \sum_{i=1}^t i a_i &= \sum_{j \leq a_1} \frac{y_j(y_j + 1)}{2} = \frac{1}{2} \sum_{j=1}^{\infty} \left( \frac{\log(1 + 1/j)}{\log 2} \right)^2 t^2 + O(t \log t) \\ &= \frac{c^2}{16(\log 2)^2} t^2 + O(t \log t). \end{aligned}$$

From the definition of  $t$ ,  $\log t = \frac{1}{2} \log_2 x - \log_3 x + O(1)$  and  $\log_2 t = \log_3 x + O(1)$ . Thus,

$$\log n \leq \left( 1 + \frac{2 \log_3 x + O(1)}{\log_2 x} \right) \left( 1 - \frac{c\varepsilon}{8 \log 2} \right)^2 \log x.$$

Hence, if  $x$  is large enough, then  $n \leq x$ . From the definition of  $n$  above, we have  $d(n) = p_1^{a_1} \cdots p_t^{a_t}$ . Therefore,

$$(2.4) \quad \begin{aligned} \log m_2(x) &\geq \log d(d(n)) = \sum_{i=1}^t \log(a_i + 1) = \sum_{j \geq 1} (y_j - y_{j+1}) \log(j + 1) = \sum_{j \geq 1} y_j \log(1 + 1/j) \\ &= \sum_{j \leq a_1} \left( \frac{\log^2(1 + 1/j)}{\log 2} t + O(1/j) \right) \\ &= \frac{c^2}{8 \log 2} t + O(\log t) \\ &= \frac{\sqrt{\log x}}{\log_2 x} \left( c + O\left( \frac{\log_3 x}{\log_2 x} \right) \right). \end{aligned}$$

□

### 3 Proof of the upper bound in Theorem 1

**Lemma 3.1.** *Let  $m_N = \min\{m : d(m) = N\}$  and write  $m_N = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ . We have*

- (i)  $\alpha_1 \geq \cdots \geq \alpha_r$ ,
- (ii)  $N' | N$  implies  $m_{N'} \leq m_N$ ,
- (iii) for each integer  $k \geq 1$ , if  $p_j > p_{r+1}^{1/2^k}$ , then  $\Omega(\alpha_j + 1) \leq k$ .

*Remark 1.* Using (2.1) and taking  $k = 1$ , we see from (iii) that if  $r$  is large, then  $\alpha_j + 1$  is prime for  $\sqrt{r} < j \leq r$ . Also, by (iii),  $\Omega(\alpha_j + 1) \ll \log_2 r$  for all  $j$ .

*Proof.* (i) This is trivial and was observed by Ramanujan [7, (32)].

(ii) If  $N' | N$ , we can find  $\alpha'_j \leq \alpha_j$  for each  $j$  such that  $N' = (\alpha'_1 + 1) \cdots (\alpha'_r + 1)$ , and clearly  $m_{N'} \leq p_1^{\alpha'_1} \cdots p_r^{\alpha'_r} \leq m_N$ .

(iii) If  $p_j > p_{r+1}^{1/2^k}$  and  $\Omega(\alpha_j + 1) > k$ , then there are integers  $a, b$  with  $\alpha_j + 1 = ab$ ,  $a \geq 2$  and  $b \geq 2^k$ . Letting

$$m^* = p_j^{b-1} p_{r+1}^{a-1} \prod_{i \neq j} p_i^{\alpha_i},$$

we see that  $d(m^*) = d(m_N) = N$ , but

$$\frac{m^*}{m_N} = p_j^{b-1-\alpha_j} p_{r+1}^{a-1} = (p_j^{-b} p_{r+1})^{a-1} < 1,$$

a contradiction. □

**Lemma 3.2.** *For every  $\varepsilon > 0$ , and for  $\omega(n) = s \geq 2$  we have*

$$d(n) \ll_{\varepsilon} \left( \frac{(2 + \varepsilon) \log n}{s \log s} \right)^s.$$

*Proof.* Write the prime factorization of  $n$  as  $n = q_1^{a_1} \cdots q_s^{a_s}$ , where  $q_1 < \cdots < q_s$ . Using the arithmetic mean - geometric mean inequality and that  $q_i \geq p_i$ , we have

$$d(n) \leq \prod_{i=1}^s (2a_i) \leq 2^s \prod_{i=1}^s (a_i \log q_i) \prod_{i=1}^s (\log p_i)^{-1} \leq \left( \frac{2 \log n}{s} \right)^s \frac{(\log s)^{\pi(s)-s}}{\log 2},$$

the last inequality coming from excluding factors corresponding to  $3 \leq p_i < s$ . Finally, the prime number theorem implies  $(\log s)^{\pi(s)} \leq (\log s)^{O(s/\log s)} \ll_{\varepsilon} (1 + \varepsilon/2)^s$ .  $\square$

**Remark.** Lemma 3.2 is fairly sharp. For example, from the inequality  $s = \omega(n) \leq (1 + o(1)) \frac{\log n}{\log_2 n}$ , and the observation that  $m_1(x)$  is monotonically increasing, we immediately obtain Wigert's upper bound for  $\log d(n)$ .

The following is the key lemma, which explains the constant  $c$ .

**Lemma 3.3.** *Let  $a_1, \dots, a_t$  be positive integers.*

(a) *we have*

$$\sum_{i=1}^t \log(a_i + 1) \leq \frac{c}{2} \left( \sum_{i=1}^t i a_i \right)^{1/2}.$$

*Moreover, the constant  $c/2$  is best possible.*

(b) *If  $a_i \geq A$  for all  $i$ , where  $A$  is a positive integer, then*

$$\sum_{i=1}^t \log(a_i + 1) \leq \left( \frac{1 + \log^2(A + 1)}{A} \sum_{i=1}^t i a_i \right)^{1/2}.$$

*Proof.* (a) Without loss of generality, suppose  $a_1 \geq \cdots \geq a_t$ . Let  $y_j = \#\{i : a_i \geq j\}$ . Then

$$\sum_{i=1}^t i a_i = \sum_{j \geq 1} \frac{y_j(y_j + 1)}{2} \geq \frac{1}{2} \sum_{j \geq 1} y_j^2.$$

By partial summation and the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{i=1}^t \log(a_i + 1) &= \sum_{j \geq 1} (y_j - y_{j+1}) \log(j + 1) = \sum_{j \geq 1} y_j \log(1 + 1/j) \\ (3.1) \qquad \qquad \qquad &\leq \left( \sum_{j \geq 1} y_j^2 \right)^{1/2} \left( \frac{c^2}{8} \right)^{1/2}. \end{aligned}$$

Moreover, the inequality in (3.1) is an equality if and only if for some real  $Y$ ,  $y_j = Y \log(1 + 1/j)$  for every  $j$ . As the  $y_j$  are integers, this cannot happen. However, we can come very close to equality in (3.1) by taking  $t$  large and choosing the  $a_i$  by (2.2), so that  $y_j = \lfloor \frac{\log(1+1/j)}{\log 2} t \rfloor$ . By (2.3) and (2.4), we have in this case

$$\sum_{i=1}^t \log(a_i + 1) = \frac{c^2}{8 \log 2} t + O(\log t), \quad \sum_{i=1}^t i a_i = \frac{c^2}{16 (\log 2)^2} t^2 + O(t \log t),$$

whence

$$\sum_{i=1}^t \log(a_i + 1) = \frac{c}{2} \left( 1 + O\left(\frac{\log t}{t}\right) \right) \left( \sum_{i=1}^t i a_i \right)^{1/2}.$$

(b) Observe that  $y_1 = y_2 = \dots = y_A$ . Arguing similarly to (3.1), we obtain

$$\begin{aligned} \sum_{i=1}^t \log(a_i + 1) &= \frac{\log(A+1)}{A} (y_1 + \dots + y_A) + \sum_{j>A} y_j \log(1 + 1/j) \\ &\leq \left( \sum_{j \geq 1} y_j^2 \right)^{1/2} \left( A \left( \frac{\log(A+1)}{A} \right)^2 + \sum_{j>A} \log^2(1 + 1/j) \right)^{1/2}. \end{aligned}$$

Observing that  $\log(1 + 1/j) < 1/j$  and  $\sum_{j>A} 1/j^2 < 1/A$ , we obtain (b).  $\square$

The next lemma is trivial.

**Lemma 3.4.** *For any positive integer  $m$ ,  $m \geq \sum_{p|m} p$ .*

*Proof of Theorem 1, upper bound.* Let  $n$  be large, let  $N = d(n)$  and factor  $N = N'N''$ , where

$$N' = u_1^{b_1} \dots u_w^{b_w}, \quad N'' = q_1^{a_1} \dots q_s^{a_s},$$

where  $u_1 < \dots < u_w$ ,  $q_1 < \dots < q_s$  are primes,  $b_i > (\log_2 n)^6$  for every  $i$  and  $a_i \leq (\log_2 n)^6$  for every  $i$ .

Write  $m_{N'} = p_1^{\beta_1} \dots p_h^{\beta_h}$ . By Lemma 3.1 (ii),  $m_{N'} \leq m_N \leq n$ , so that  $h \ll \log n$ . By Lemma 3.1 (iii),  $\Omega(\beta_i + 1) \ll \log_2 h \ll \log_3 n$  for every  $i$ . Since  $d(m_{N'}) = (\beta_1 + 1) \dots (\beta_h + 1) = N'$ , for each  $j \leq h$  there are  $\gg \frac{b_j}{\log_3 n}$  values of  $i$  for which  $u_j | (\beta_i + 1)$ . Thus, by Lemma 3.4,

$$\begin{aligned} \log n &\geq \log m_{N'} \geq (\log 2) \sum_{i=1}^h \beta_i \geq \frac{\log 2}{2} \sum_{i=1}^h (\beta_i + 1) \\ &\geq \frac{\log 2}{2} \sum_{i=1}^h \sum_{p | (\beta_i + 1)} p \gg \sum_{j=1}^w u_j \frac{b_j}{\log_3 n} \geq \frac{1}{\log_3 n} \sum_{j=1}^w j b_j. \end{aligned}$$

Combining this estimate with Lemma 3.3 (b) with  $A = (\log_2 n)^6$  gives

$$(3.2) \quad \log d(N') = \sum_{j=1}^w \log(b_j + 1) \ll \frac{\log_3 n}{(\log_2 n)^3} \left( \sum_{j=1}^w j b_j \right)^{1/2} \ll \frac{(\log n)^{1/2} (\log_3 n)^{3/2}}{(\log_2 n)^3}.$$

Next, we bound  $d(N'')$ .

Case 1) If  $s \leq \frac{(\log n)^{1/2}}{(\log_2 n)^3}$ , Lemma 3.2 implies that  $\log d(N'') \ll \frac{(\log n)^{1/2}}{(\log_2 n)^2}$ .

Case 2) Now suppose that  $s > \frac{(\log n)^{1/2}}{(\log_2 n)^3}$ . Write  $m_{N''} = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ . By Lemma 3.1 (iii),

$$(3.3) \quad r \leq \Omega(N'') = \sum_{j=1}^s a_j = \sum_{i=1}^r \Omega(\alpha_i + 1) \leq r + \sum_{k \geq 2} \pi(p_{r+1}^{1/2^k}) = r + O((r/\log r)^{1/2}).$$

In particular,  $r + O((r/\log r)^{1/2}) \geq a_1 + \dots + a_s \geq s$ , so  $r \gg s > \frac{(\log n)^{1/2}}{(\log_2 n)^3}$ . Thus, for large enough  $n$ ,  $a_1 + \dots + a_s \leq r + \sqrt{r}$ . Also by Lemma 3.1 (iii),  $\alpha_j + 1$  is prime for  $j > \sqrt{r}$ . Let  $\varepsilon = 20 \frac{\log_3 n}{\log_2 n}$ . By the lower bound on  $s$ , and using  $a_i \leq (\log_2 n)^6$ ,

$$(3.4) \quad \sum_{j > s - s^{1-\varepsilon}} a_j \geq s^{1-\varepsilon} \geq 2 (s (\log_2 n)^6)^{1/2} \geq 2 (\Omega(N''))^{1/2} \geq 2\sqrt{r},$$

hence, using (3.3),

$$\sum_{j \leq s - s^{1-\varepsilon}} a_j \leq \Omega(N'') - 2\sqrt{r} \leq r - \sqrt{r}.$$

Using Lemma 3.1 (i),  $\alpha_i + 1 = q_1$  for  $r - a_1 < i \leq r$ , and similarly for each  $j \leq s - s^{1-\varepsilon}$ ,  $\alpha_i + 1 = q_j$  for  $r - (a_1 + \cdots + a_j) < i \leq r - (a_1 + \cdots + a_{j-1})$ . We obtain

$$\begin{aligned} \log m_{N''} &\geq \sum_{\sqrt{r} < i \leq r} \alpha_i \log p_i \geq \sum_{j \leq s - s^{1-\varepsilon}} (q_j - 1) \sum_{m=r-(a_1+\cdots+a_j)+1}^{r-(a_1+\cdots+a_{j-1})} \log p_m \\ &\geq \sum_{j \leq s - s^{1-\varepsilon}} (p_j - 1) a_j \log(r - (a_1 + \cdots + a_j)). \end{aligned}$$

By (3.4), uniformly for  $j \leq s - s^{1-\varepsilon}$  we have

$$r - (a_1 + \cdots + a_j) = r - \Omega(N'') + a_{j+1} + \cdots + a_s \geq s - j - \sqrt{r} \geq \frac{1}{2} s^{1-\varepsilon}.$$

Using (2.1),  $p_j \geq j \log j + 1$  for large  $j$ . Hence, by Lemma 3.1 (ii),

$$\begin{aligned} \log n &\geq \log m_{N''} \geq \sum_{s^{1-\varepsilon} \leq j \leq s - s^{1-\varepsilon}} (j \log j) a_j (\log s + O(\log_3 n)) \\ &\geq (1 + O(\varepsilon)) \frac{(\log_2 n)^2}{4} \sum_{s^{1-\varepsilon} \leq j \leq s - s^{1-\varepsilon}} j a_j. \end{aligned}$$

By the definition of  $\varepsilon$ ,  $s^\varepsilon \gg (\log_2 n)^9$ . Also, trivially  $\sum_{j=1}^s j a_j \geq 1 + 2 + \cdots + s \geq \frac{1}{2} s^2$ . Recalling that  $a_j \leq (\log_2 n)^6$  for every  $j$ , we have

$$\begin{aligned} \sum_{s^{1-\varepsilon} \leq j \leq s - s^{1-\varepsilon}} j a_j &= \sum_{j=1}^s j a_j + O(s^{2-\varepsilon} (\log_2 n)^6) = \sum_{j=1}^s j a_j + O(s^2 (\log_2 n)^{-3}) \\ &= (1 + O(1/\log_2 n)) \sum_{j=1}^s j a_j. \end{aligned}$$

Combining the last two inequalities gives

$$\log n \geq \left(1 + O\left(\frac{\log_3 n}{\log_2 n}\right)\right) \frac{(\log_2 n)^2}{4} \sum_{j=1}^s j a_j.$$

Applying Lemma 3.3 (a), we conclude that

$$(3.5) \quad \log d(N'') = \sum_{j=1}^s \log(a_j + 1) \leq \frac{c}{2} \left(\sum_{j=1}^s j a_j\right)^{1/2} \leq c \frac{\sqrt{\log n}}{\log_2 n} \left(1 + O\left(\frac{\log_3 n}{\log_2 n}\right)\right).$$

Recall that we have a smaller upper bound for  $\log d(N'')$  in case 1). Finally, using  $d(d(n)) = d(N')d(N'')$  and combining (3.2) and (3.5), we obtain the desired upper bound for  $d(d(n))$ .  $\square$

## 4 Proof of Theorem 2

*Proof of Theorem 2.* For the lower bound, let  $x$  be large and put  $n = \prod_{i=1}^s p_i^{p_i-1}$ , where  $s$  is the largest integer such that  $n \leq x$ . Recall that  $p_j$  is the  $j$ -th smallest prime. Then  $d(n) = \prod_{i=1}^s p_i$ , thus  $\omega(d(n)) = s$ . By (2.1),

$$\log n = \sum_{i=1}^s (p_i - 1) \log p_i = \sum_{i=1}^s i \log^2 i + O(i \log i \log_2 i) = \frac{1}{2} s^2 \log^2 s + O(s^2 \log s \log_2 s).$$

Solving for  $s$  gives  $s = \frac{\sqrt{8 \log n}}{\log_2 n} + O\left(\frac{\sqrt{\log n \log_3 n}}{\log_2^2 n}\right)$ . We now prove a lower bound on  $n$ . Since  $p_{s+1} \sim s \log s \sim \sqrt{2 \log n} \ll \sqrt{\log x}$  by (2.1), we have

$$x \geq n \geq x p_{s+1}^{-p_{s+1}} = x \exp\left(-O\left(\sqrt{\log x} \log_2 x\right)\right).$$

That is,  $\log n = \log x + O(\sqrt{\log x} \log_2 x)$ . Therefore,  $s = \frac{\sqrt{8 \log x}}{\log_2 x} + O\left(\frac{\sqrt{\log x \log_3 x}}{\log_2^2 x}\right)$ .

Now let  $n$  be a large, positive integer factored as  $n = n_1 n_2$ ,  $n_1 = \prod_{i=1}^r q_i^{a_i}$ ,  $n_2 = \prod_{i=1}^{r'} (q'_i)^{a'_i}$ , where  $q_i, q'_i$  are primes,  $q_i > P$  and  $q'_i \leq P$  for each  $i$ , where  $P = \frac{\sqrt{\log n}}{\log_2 n}$ . We have

$$(4.1) \quad \omega(d(n)) \leq \omega(d(n_1)) + \omega(d(n_2)).$$

Since  $\omega(n_2) \leq \pi(P) \ll \frac{\sqrt{\log n}}{(\log_2 n)^2}$ , Lemma 3.2 implies  $\log d(n_2) \ll \sqrt{\log n} / \log_2 n$ . Applying the elementary inequality  $\omega(u) \ll \frac{\log u}{\log_2 u}$  gives

$$(4.2) \quad \omega(d(n_2)) \ll \frac{\sqrt{\log n}}{(\log_2 n)^2}.$$

Next,

$$\log n_1 \geq (\log P) \sum_{i=1}^r a_i = \left(\frac{\log_2 n}{2} - \log_3 n\right) \sum_{i=1}^r a_i.$$

Letting  $s = \omega(d(n_1)) = \omega(\prod (a_i + 1))$ , Lemma 3.4 implies that

$$\sum_{i=1}^r a_i \geq \sum_{i=1}^r \sum_{p|(a_i+1)} (p-1) \geq \sum_{i=1}^s (p_i - 1) \geq \sum_{i=1}^s (i \log i + O(1)) = \frac{1}{2} s^2 \log s + O(s^2).$$

Here we used the one-sided inequality  $p_i \geq i \log i + O(1)$  deduced from (2.1). Thus,

$$\log n \geq \log n_1 \geq \left(\frac{1}{4} + O\left(\frac{\log_3 n}{\log_2 n}\right)\right) (\log_2 n) s^2 \log s + O(s^2 \log_2 n).$$

Consider two cases: (i)  $s \leq \frac{\sqrt{\log n}}{\log_2 n}$ , (ii)  $s > \frac{\sqrt{\log n}}{\log_2 n}$ . In case (ii), we have  $\frac{\log n}{\log_2^2 n} \geq \left(\frac{1}{8} + O\left(\frac{\log_3 n}{\log_2 n}\right)\right) s^2$ , and we obtain in both cases

$$\omega(d(n_1)) = s \leq \frac{\sqrt{8 \log n}}{\log_2 n} + O\left(\frac{\sqrt{\log n \log_3 n}}{\log_2^2 n}\right),$$

Combining this inequality with (4.1) and (4.2), we obtain the desired upper bound for  $\omega(d(n))$ .  $\square$

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